ON THE CLASSIFICATION OF PLANAR CONTACT STRUCTURES

M. FIRAT ARIKAN AND SELAHI DURUSOY

Abstract. In this paper, we focus on contact structures supported by planar open book decompositions. We study right-veering diffeomorphisms to keep track of overtwistedness property of contact structures under some monodromy changes. As an application we give infinitely many examples of overtwisted contact structures supported by open books whose pages are the four-punctured sphere, and also we prove that a certain family is holomorphically fillable using lantern relation.

1. Introduction

Let (M, ξ) be a closed oriented 3-manifold with the contact structure ξ, and let (S, h) be an open book (decomposition) of M which is compatible with ξ. (We refer the reader to [Et3, Ge] for contact geometry, and to [Et2, Gd] for open books and compatibility). Based on the correspondence given in [Gi], two topological invariants were defined in [EO]:

$$sg(\xi) = \min\{\ g(S) \mid (S, h) \text{ an open book supporting } \xi\},$$

called the *support genus* of ξ, and

$$bn(\xi) = \min\{\ |\partial S| \mid (S, h) \text{ an open book supporting } \xi \text{ and } g(S) = sg(\xi)\},$$

called the *binding number* of ξ. It is proved in [Et1] that if (M, ξ) is overtwisted, then $sg(\xi) = 0$, i.e., ξ is supported by a planar open book. The algorithm given in [Ar1] finds a reasonable upper bound for $sg(\xi)$ using the given contact surgery diagram of ξ. On the other hand, for $sg(\xi) = 0$ and $bn(\xi) \leq 2$, the complete list of all such planar contact structures (up to isotopy) is given in [EO]. The case when $sg(\xi) = 0$ and $bn(\xi) = 3$ is classified in [Ar2]. In the last two mentioned papers, it was also shown which structures are tight and which ones are overtwisted. As an application of the techniques which we will develop, we will partially consider the case where $sg(\xi) = 0$ and $bn(\xi) \leq 4$ at the end of this paper. We refer the reader [L] for tight contact structures supported by four-punctured sphere.

The structure of this paper is as follows: In Section 1, we state the theorems that we will prove later in the paper. After the preliminaries (Section 2), right-veering diffeomorphisms and overtwisted planar contact structures are studied in Section 3 where we also give alternative proofs of some results recently proved in [Y] (see Lemma 3.3, Remark 3.5, Lemma 3.6). In Section 4, we focus on the four-punctured sphere and prove our main results.

Let D_γ denote the right Dehn twist along the simple closed curve γ. Most of the time we'll write γ instead D_γ for simplicity. For any bordered surface S, let $Aut(S, \partial S)$ be the group of isotopy classes of orientation preserving diffeomorphisms of S which restrict to identity on ∂S. In $Aut(S, \partial S)$, we will multiply a new element from the right of the previously given word although we compose the corresponding difeomorphisms of S from left.

For a given fixed open book (S, h) of a 3-manifold M, by [Gi], (S, h) determines a unique contact manifold $(M_{(S,h)}, \xi_{(S,h)})$ up to contactomorphism. We will shorten the notation as (M_h, ξ_h) if the surface S is clear from the content.

Let Σ be the four$-$punctured sphere obtained by deleting the interiors of four disks from the 2$-$sphere S^2 (see Figure 1). Let C_1, C_2, C_3, C_4 be the boundary components of Σ, and let

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a, b, c, d denote the simple closed curves parallel to the boundary components C_1, C_2, C_3, C_4, respectively. Also consider the simple closed curves e, f, g, h in Σ given as in Figure 1.

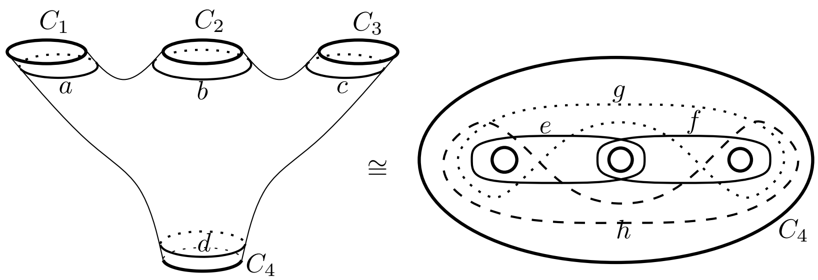

FIGURE 1. Four−punctured sphere Σ, and the simple closed curves.

Let $\phi \in Aut(\Sigma, \partial\Sigma)$ be any element. In Section 4, it will be clear that we can write

$$\phi = a^{r_1} b^{r_2} c^{r_3} d^{r_4} e^{m_1} f^{n_1} \cdots e^{m_s} f^{n_s}$$

for some integers m_i's and n_i's (see Lemma 4.2). Our main results are the following:

Theorem 1.1. *The contact manifold* (M_ϕ, ξ_ϕ) *is holomorphically fillable in each of the following cases:*

(H1) $s = 1, max\{m_1, n_1\} \geq 0, min\{r_k\} \geq max\{-m_1, -n_1, 0\}$,
(H2) $s = 1, m_1 < 0, n_1 < 0, max\{m_1, n_1\} = -1, min\{r_k\} \geq -m_1 - n_1 - 1$,
(H3) $s = 1, m_1 < 0, n_1 < 0, max\{m_1, n_1\} < -1, min\{r_k\} \geq -m_1 - n_1 - 2$,
(H4) $s > 1, min\{r_k\} \geq \sum_{i=1}^{s} max\{-m_i, 0\} + \sum_{j=1}^{s} max\{-n_j, 0\}$.

For the other results, we focus only on the elements of the form $\phi = a^{r_1} b^{r_2} c^{r_3} d^{r_4} e^{m_1} f^{n} e^{m_2}$ or $\phi = a^{r_1} b^{r_2} c^{r_3} d^{r_4} f^{n_1} e^{m} f^{n_2}$. Note that it is enough to study only one of these forms because of the symmetry between e and f given by rotation, so we will consider only the first one.

Theorem 1.2. *The contact structure* ξ_ϕ *is overtwisted in the following cases:*

(OT1) $r_k < 0$ *for some* k,
(OT2) $r_k = 0$ *for some* k *and* $min\{m, n\} < 0$,
(OT3) $min\{r_k\} = 1$, $\{r_2 = 1$ *or* $r_4 = 1\}$, $min\{m, n\} < 0$ *and* $mn \geq 2$,
(OT4) $min\{r_k\} = 1$, $\{r_1 = 1$ *or* $r_3 = 1\}$, $min\{m, n\} < 0$ *and* $mn \geq 2$,

where $\phi = a^{r_1} b^{r_2} c^{r_3} d^{r_4} e^{m_1} f^{n} e^{m_2} \in Aut(\Sigma, \partial\Sigma)$ *and* $m = m_1 + m_2$.

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2. Preliminaries

First, we state the following classical fact which will be used in Section 4. We also give a proof since the authors couldn't find the given version of the theorem in the literature.

Theorem 2.1. *Let S be any surface with nonempty boundary, and let $\sigma, h \in Aut(S, \partial S)$. Then there exists a contactomorphism*

$$(M_{(S,h)}, \xi_{(S,h)}) \cong (M'_{(S,\sigma h \sigma^{-1})}, \xi'_{(S,\sigma h \sigma^{-1})}).$$

Proof. The proof based on idea of breaking up the monodromy $\sigma h \sigma^{-1}$ into pieces as depicted in Figure 2. First take each glued solid torus (around each binding component) out from both $(M_{(S,h)}, \xi_{(S,h)})$ and $(M'_{(S,\sigma h \sigma^{-1})}, \xi'_{(S,\sigma h \sigma^{-1})})$ to get the mapping tori $S(h)$ and $S(\sigma h \sigma^{-1})$.

By breaking the monodromy $\sigma h \sigma^{-1}$, the mapping torus $S(\sigma h \sigma^{-1}) = [0,1] \times S/(1,x) \sim (0, \sigma h \sigma^{-1}(x))$ can be constructed also as follows: We write

$$S(\sigma h \sigma^{-1}) = (\coprod_{i=1}^{4} S_i)/\sim,$$

where $S_i = S \times [\frac{i-1}{4}, \frac{i}{4}]$ and $\sim$ is the equivalence relation that glues $S \times \{\frac{1}{4}\}$ in S_1 to $S \times \{\frac{1}{4}\}$ in S_2 by σ, glues $S \times \{\frac{1}{2}\}$ in S_2 to $S \times \{\frac{1}{2}\}$ in S_3 by h, glues $S \times \{\frac{3}{4}\}$ in S_3 to $S \times \{\frac{3}{4}\}$ in S_4 by σ^{-1}, glues $S \times \{1\}$ in S_4 to $S \times \{0\}$ in S_1 by the identity map id. (See the picture on the left in Figure 2.)

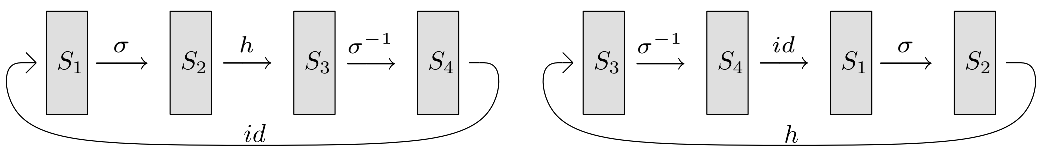

FIGURE 2. Mapping torus $S(\sigma h \sigma^{-1})$, before and after the cyclic permutation.

Since $S(\sigma h \sigma^{-1})$ is a fiber bundle over the circle S^1, we are free to change its monodromy by any cyclic permutation. Therefore, the monodromy element $\sigma^{-1} \cdot id \cdot \sigma h = h$ also gives the same fiber bundle $S(\sigma h \sigma^{-1})$ (the picture on the right in Figure 2 shows the new configuration of $S(\sigma h \sigma^{-1})$ after the cyclic permutation). Therefore, $S(h) = S(\sigma h \sigma^{-1})$. By gluing all solid tori back using identity, we conclude that $(M_{(S,h)}, \xi_{(S,h)})$ is contactomorphic to $(M'_{(S,\sigma h \sigma^{-1})}, \xi'_{(S,\sigma h \sigma^{-1})})$. $\square$

A *Stein manifold* of dimension four is a triple (X^4, J, ψ) where J is a complex structure on X, $\psi : X \to \mathbb{R}$, and the 2–form $\omega_\psi = -d(d\psi \circ J)$ is non-degenerate. We say that (M^3, ξ) is *Stein (holomorphically) fillable* if there is a Stein manifold (X^4, J, ψ) such that ψ is bounded from below, M is a non-critical level of ψ, and $-(d\psi \circ J)$ is a contact form for ξ. The following fact was first implied in [LP], and then in [AO]. The version given below is due to Giroux and Matveyev. For a proof, see [OSt].

Theorem 2.2. *A contact structure ξ on M^3 is holomorphically fillable if and only if ξ is supported by some open book whose monodromy admits a factorization into positive Dehn twists only.*

Right-veering Diffeomorphisms: We recall the right-veering diffeomorphisms originally introduced in [HKM1]. If S is a compact oriented surface with $\partial S \neq \emptyset$, the submonoid $Veer(S, \partial S)$ of right-veering elements in $Aut(S, \partial S)$ is defined as follows: Let α and β be isotopy classes (relative to the endpoints) of properly embedded oriented arcs $[0,1] \to S$ with a common initial point $\alpha(0) = \beta(0) = x \in \partial S$. Let $\pi : \tilde{S} \to S$ be the universal cover of S (the interior of $\tilde{S}$ will always be $\mathbb{R}^2$ since S has at least one boundary component), and let $\tilde{x} \in \partial \tilde{S}$ be a lift of $x \in \partial S$. Take lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β with $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}$. $\tilde{\alpha}$ divides $\tilde{S}$ into two regions – the region "to the left" and the region "to the right". We say that β is *to the right* of α, denoted $\alpha \geq \beta$, if either $\alpha = \beta$ (and hence $\tilde{\alpha}(1) = \tilde{\beta}(1)$), or $\tilde{\beta}(1)$ is in the region to the right (Figure 3).

As an alternative way to passing to the universal cover, we first isotope α and β, while fixing their endpoints, so that they intersect transversely (including at the endpoints) and with the fewest possible number of intersections. Then β is to the right of α if the tangent vectors $(\dot{\beta}(0), \dot{\alpha}(0))$ define the orientation on S at x.

Definition 2.3 ([HKM1])**.** *Let $h : S \to S$ be a diffeomorphism that restricts to the identity map on ∂S. Let α be a properly embedded oriented arc starting at a basepoint $x \in \partial S$. Then*

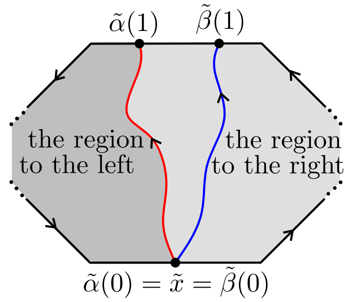

FIGURE 3. Lifts of α and β in the universal cover $\tilde{S}$.

h is right-veering *(that is, $h \in Veer(S, \partial S)$) if for every choice of basepoint $x \in \partial S$ and every choice of α based at x, $h(\alpha)$ is to the right of α (at x). If C is a boundary component of S, we say is h is* right-veering with respect to C *if $h(\alpha)$ is to the right of α for all α starting at a point on C.*

It turns out that $Veer(S, \partial S)$ is a submonoid and we have the inclusions:

$$Dehn^+(S, \partial S) \subset Veer(S, \partial S) \subset Aut(S, \partial S).$$

We will use the following two results of [HKM1].

Theorem 2.4 ([HKM1])**.** *A contact structure (M, ξ) is tight if and only if all of its compatible open book decompositions (S, h) have right-veering $h \in Veer(S, \partial S) \subset Aut(S, \partial S)$.*

Lemma 2.5 ([HKM1])**.** *Let S be a hyperbolic surface with geodesic boundary and $\gamma \in Aut(S, \partial S)$. Let $S' \subsetneq S$ be a subsurface, also with geodesic boundary, and let C be a common boundary component of S and S'. If γ is the identity map when restricted to S', δ is a closed curve parallel to and disjoint from C, and m is a positive integer, then $D_\delta^m \cdot \gamma$ is right-veering with respect to C.*

Remark 2.6. This lemma is useful in proving right-veering property and will be used in the proof of Theorem 4.3. Note that in the case of four$-$punctured sphere, S' must have at least two holes to apply Lemma 2.5

3. Right-Veering Diffeomorphisms and Overtwisted contact structures

In this section we will give the results which will be used to prove Theorem 1.2.

Lemma 3.1. *Let S be a planar hyperbolic surface with geodesic boundary $\partial S = \cup_{i=1}^{l} C_i$, $l \geq 4$. Suppose $h \in Aut(S, \partial S)$ and there is a properly embedded arc γ starting at $x \in C_i$, ending at C_j such that $h(\gamma)$ is to the left of γ at x and $i \neq j$. Then $(h \cdot D_\delta)(\gamma)$ is to the left of γ at $x \in C_i$ for any curve δ parallel to C_k with $k \neq i$.*

Proof. Isotoping if necessary, we may assume that γ and $h(\gamma)$ intersect minimally. We need to analyze two cases:
Case 1. Suppose $k \neq j$. Then we may assume $\gamma \cap \delta = \emptyset$, and so $h(\gamma) \cap \delta = \emptyset$. That is, D_δ fixes both γ and $h(\gamma)$. This implies that $D_\delta(h(\gamma)) = h(\gamma)$ is to the left of γ.
Case 2. Suppose $k = j$. First note that $h \neq id_S$ since h is not right-veering. Therefore, there exists a region $R \subset S$ such that

(1) R is an embedded disk punctured $r-$times for some $0 < r < m - 2$, and
(2) $\partial R \subset \gamma \cup h(\gamma) \cup \partial S$.

Let $C_{i_1}, \cdots, C_{i_r}$ be the common components of ∂S and ∂R. We may assume that ∂R contains the common initial point x and the first intersection point y (of γ and $h(\gamma)$) coming right after x (See Figure 4).

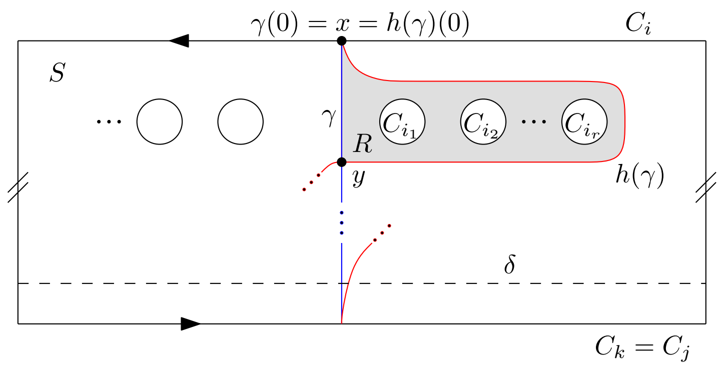

FIGURE 4. $h(\gamma)$ is to the left of γ (left and right sides are identified).

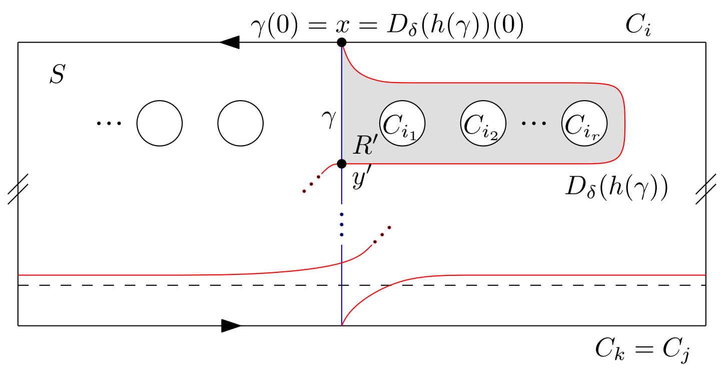

FIGURE 5. $D_\delta(h(\gamma))$ is to the left of γ (left and right sides are identified).

Since the Dehn twist D_δ is isotopic to the identity outside of a small neighborhood of δ, the image $R' = D_\delta(R)$ is isotopic to R. In particular, $\partial R' \cap D_\delta(h(\gamma))$ is to the left of $\partial R' \cap \gamma$ (see Figure 5). Note that $D_\delta(h(\gamma))$ and γ are also intersecting minimally. Therefore, we conclude that $(h \cdot D_\delta)(\gamma) = D_\delta(h(\gamma))$ is to the left of γ. □

The following corollary of Lemma 3.1 is immediate with the help of Theorem 2.4.

Corollary 3.2. *Let S be a planar hyperbolic surface with geodesic boundary $\partial S = \cup_{i=1}^{l} C_i$, $l \geq 4$. Suppose $h \in Aut(S, \partial S)$ is not right veering with respect to C_i for some i, and so the contact structure $\xi_{(S,h)}$ is overtwisted. Then the contact structure $\xi_{(S,h\cdot {D_\delta}^k)}$ is also overtwisted for any $k \in \mathbb{Z}_+$ and for any curve δ parallel to the boundary component which is different than C_i.* □

Let us now interpret the notion of right-veering in terms of the circle at infinity as in [HKM1]. Let S be any hyperbolic surface with geodesic boundary ∂S. The universal cover $\pi : \tilde{S} \to S$ can be viewed as a subset of the Poincaré disk $D^2 = \mathbb{H}^2 \cup S^1_\infty$. Let C be a component of ∂S and L be a component of $\pi^{-1}(C)$. If $h \in Aut(S, \partial S)$, let $\tilde{h}$ be the lift of h that is the identity on L. The closure of $\tilde{S}$ in D^2 is a starlike disk. L is contained in $\partial \tilde{S}$. Denote its complement in $\partial \tilde{S}$ by L_∞. Orient L_∞ using the boundary orientation of $\tilde{S}$ and then linearly order the interval L_∞ via an orientation-preserving homeomorphism with $\mathbb{R}$. The lift $\tilde{h}$ induces a homeomorphism $h_\infty : L_\infty \to L_\infty$. Also, given two elements a, b in $Homeo^+(\mathbb{R})$, the group of orientation-preserving homeomorphisms of $\mathbb{R}$, we write $a \geq b$ if $a(z) \geq b(z)$ for all $z \in \mathbb{R}$ and $a > b$ if $a(z) > b(z)$ for all $z \in \mathbb{R}$. In this setting, an element h is right-veering with respect to C if $id \geq h_\infty$. Equivalently, if α is any properly embedded

curve starting at a point $\alpha(0) \in C$, and $\tilde{\alpha}$ is the lift of α starting at the lift $\tilde{\alpha}(0) \in L$ of x, then we have

$$h(\alpha) \text{ is to the right of } \alpha \iff \tilde{\alpha}(1) \geq h_\infty(\tilde{\alpha}(1))$$

Therefore, h is not right-veering with respect to C if there is an arc α starting at C such that we have $\tilde{\alpha}(1) < h_\infty(\tilde{\alpha}(1))$.

Lemma 3.3. *Let S be any hyperbolic surface with geodesic boundary ∂S. Suppose $h \in Aut(S, \partial S)$ and there is a properly embedded arc γ starting at $x \in C \subset \partial S$ such that $h(\gamma)$ is to the left of γ at x. Then $(h \cdot D_\sigma^{-1})(\gamma)$ is to the left of γ at $x \in C$ for any simple closed curve σ in S.*

Proof. Write σ for D_σ. Fix the identification of L_∞ with $\mathbb{R}$ as above. Consider the lift $\tilde{\gamma}$ and induced homeomorphisms $h_\infty, \sigma_\infty, \sigma_\infty^{-1} : L_\infty \to L_\infty$. Since $\sigma^{-1} \cdot \sigma = id_S$, we have

$$(\sigma^{-1} \cdot \sigma)_\infty = \sigma_\infty \circ \sigma_\infty^{-1} = (id_S)_\infty.$$

Therefore, σ_∞^{-1} must map any point in L_∞ to its left because σ is right-veering. In particular, $(h \cdot \sigma^{-1})_\infty(\tilde{\gamma}(1)) = \sigma_\infty^{-1}(h_\infty(\tilde{\gamma}(1)))$ is to the left of $h_\infty(\tilde{\gamma}(1))$ which is (by the assumption) to the left of $\tilde{\gamma}(1)$. That is, $(h \cdot \sigma^{-1})_\infty(\tilde{\gamma}(1)) > h_\infty(\tilde{\gamma}(1)) > \tilde{\gamma}(1)$. □

Corollary 3.4. *Let S be a planar hyperbolic surface with geodesic boundary $\partial S = \cup_{i=1}^l C_i$, $l \geq 4$. Suppose $h \in Aut(S, \partial S)$ is not right veering with respect to C_i for some i, and so the contact structure $\xi_{(S,h)}$ is overtwisted. Then the contact structure $\xi_{(S,h \cdot D_\sigma{}^k)}$ is also overtwisted for any $k \in \mathbb{Z}_-$ and for any simple closed curve σ in Σ.* □

Remark 3.5. The idea used in the proof of Lemma 3.3 gives a simple proof for Lemma 6 of [Y]. Moreover, the following lemma is given as Lemma 5 in [Y]. We want to give a different proof for it using the idea of the circle at infinity.

Lemma 3.6. *Let S be a hyperbolic surface with geodesic boundary, and let $h \in Aut(S, \partial S)$ be a right-veering diffeomorphism. Then $h' = \sigma h \sigma^{-1}$ is right-veering for any $\sigma \in Aut(S, \partial S)$.*

Proof. Clearly, it is enough to consider the case when σ is a single Dehn twist. First, assume that σ is a positive Dehn twist. We need to show that h' is right-veering with respect to any boundary component of S. We will use the notations introduced in the previous paragraph. So fix the boundary component C, and an identification of L_∞ with $\mathbb{R}$ as above. Let α be any properly embedded curve in S starting at a point $\alpha(0) \in C$. Consider the lift $\tilde{\alpha}$ and induced homeomorphisms $h'_\infty, h_\infty, \sigma_\infty, \sigma_\infty^{-1} : L_\infty \to L_\infty$. From their definitions we have

$$h'_\infty(\tilde{\alpha}(1)) = \tilde{h'}(\tilde{\alpha}(1)) = \widetilde{\sigma h \sigma^{-1}}(\tilde{\alpha}(1)) = \tilde{\sigma}\tilde{h}\widetilde{\sigma^{-1}}(\tilde{\alpha}(1)) = \sigma_\infty h_\infty \sigma_\infty^{-1}(\tilde{\alpha}(1))$$

Suppose that $\sigma_\infty^{-1}(\tilde{\alpha}(1)) = a \in L_\infty$ and $h_\infty(a) = b \in L_\infty$. Then since

$$\sigma_\infty(b) = ((\sigma^{-1})^{-1})_\infty(b) = (\sigma_\infty^{-1})^{-1}(b),$$

b must be mapped (by σ_∞) to a point in L_∞ which is to the right of $\tilde{\alpha}(1)$ as we illustrated in Figure 6.

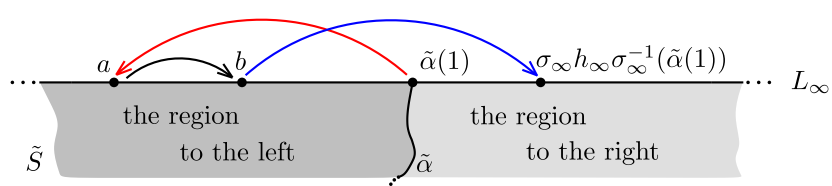

FIGURE 6. The point $\tilde{\alpha}(1) \in L_\infty \approx \mathbb{R}$, and how it is mapped to the right of itself.

Equivalently, $\tilde{\alpha}(1) \geq \sigma_\infty h_\infty \sigma_\infty^{-1}(\tilde{\alpha}(1)) = h'_\infty(\tilde{\alpha}(1))$ implying that h' is right-veering with respect to C. The proof of the case when σ is a negative Dehn twist uses exactly the same argument, so we omit it. □

4. Four–Punctured Sphere and the Proofs of Main Theorems

For simplicity, we will denote the Dehn twist along any simple closed curve by the same letter we use for that curve.

Definition 4.1. *A representative of an element $\phi \in Aut(\Sigma, \partial\Sigma)$ is said to be in* ***reduced form*** *if s is the smallest integer such that ϕ can be written as*

$$\phi = a^{r_1} b^{r_2} c^{r_3} d^{r_4} e^{m_1} f^{n_1} e^{m_2} f^{n_2} \cdots e^{m_{s-1}} f^{n_{s-1}} e^{m_s} f^{n_s}$$

where r_k, m_i, n_i are all integer for $1 \le k \le 4$, $1 \le i \le s$ with possibly m_1 or n_s zero.

Lemma 4.2. *Any element $\phi \in Aut(\Sigma, \partial\Sigma)$ can be written in reduced form.*

Proof. From braid group representation of full mapping class group, we know that the mapping class group $Aut(\Sigma, \partial\Sigma)$ can be generated by Dehn twists along the simple closed curves a, b, c, d, e, f, g, h given in Figure 1 (see [Bi] for details). Therefore, any element ϕ of $Aut(\Sigma, \partial\Sigma)$ can be written as a word consisting of only a, b, c, d, e, f, g, h and their inverses. Since a, b, c, d, are in the center of $Aut(\Sigma, \partial\Sigma)$, we can bring them to any position we want. For the second part including e and f, we use the well-known *lantern relation* (also known as *4-holed sphere relation*). In terms of our generators we will use two different lantern relations. Namely, we have

$$gef = abcd \qquad \text{and} \qquad hfe = abcd.$$

These give $g = abcdf^{-1}e^{-1}$ and $h = abcde^{-1}f^{-1}$. Therefore, we can exchange any power of g and h in the word defining ϕ by some products of $a, b, c, d, e^{-1}, f^{-1}$ (and $a^{-1}, b^{-1}, c^{-1}, d^{-1}, e, f$ for negative powers of g and h). Combining (and canceling if there is any) the powers of e and f, and commuting the generators a, b, c, d, we get the reduced form of ϕ as claimed. □

We first prove Theorem 1.1 using the lantern relations.

Proof of Theorem 1.1. Let $\phi = a^{r_1} b^{r_2} c^{r_3} d^{r_4} e^{m_1} f^{n_1} \cdots e^{m_s} f^{n_s} \in Aut(\Sigma, \partial\Sigma)$. We will show how to obtain a monodromy for the same open book which is a product of positive Dehn twists and use Theorem 2.2. Using lantern relations, we can replace each e^{-1} by $a^{-1}b^{-1}c^{-1}d^{-1}hf$ and each f^{-1} by $a^{-1}b^{-1}c^{-1}d^{-1}ge$. This proves (H1) and (H4). In the case where $s = 1$, we can use fewer lantern relations by first doing the following: using the lantern relations, replace $e^{-1}f^{-1}$ by $a^{-1}b^{-1}c^{-1}d^{-1}h$. Moreover, if $max\{m_1, n_1\} < -1$, also use the lantern relations to replace $e^{-}1$ by $a^{-1}b^{-1}c^{-1}d^{-1}fg$, and use Theorem 2.1 to cancel the new initial f with the last f^{-1}. □

We note that the two simplifications mentioned in the case of $s = 1$ can also be applied in general, but negative powers of e and f need not be adjacent (even after a cyclic permutation). We also remark that changing the order of the products of e and f can result in not only different contact manifolds, but also topologically different manifolds. For instance, $\phi = e^2 f^2$ and $\phi = efef$ are not conjugate to each other, and the underlying topological manifolds are not diffeomorphic (see Figure 7 for their surgery diagrams).

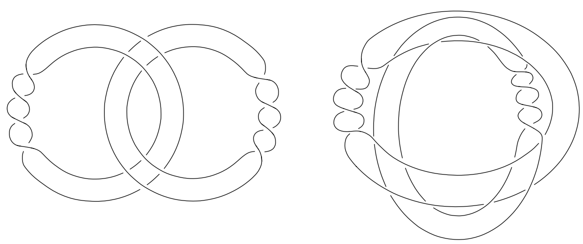

Figure 7. Diagrams for $\phi = e^2 f^2$ and $\phi = efef$ (all coefficients are -3).

Next, we characterize the overtwisted structures stated in the introduction.

Proof of Theorem 1.2. By using Theorem 2.1, we will prove the statements for $\xi_{\phi'}$ where $\phi' = a^{r_1}b^{r_2}c^{r_3}d^{r_4}e^m f^n$. To prove (OT1), consider the properly embedded curves $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ starting at the boundary components C_1, C_2, C_3, C_4, respectively, and their images under ϕ' as given in Figure 8. In all the pictures, we are assuming $m > 0$, $n > 0$, and $r_k = -1$

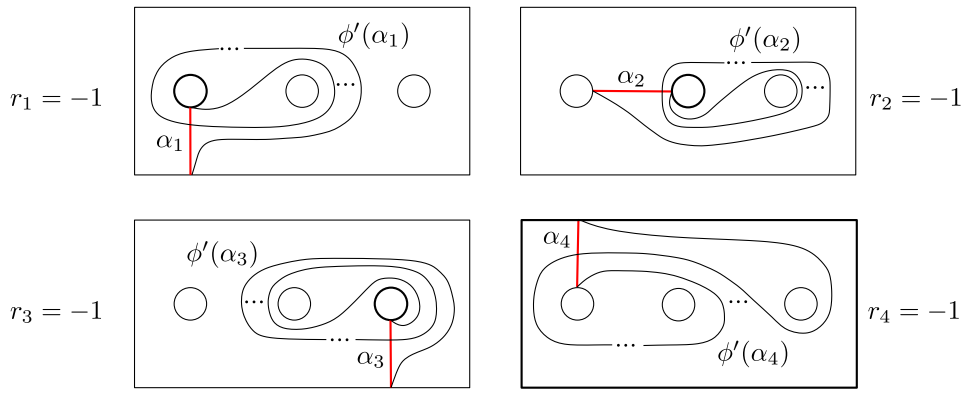

FIGURE 8. The curves α_k and their images under ϕ' in Σ.

(otherwise the fact that ϕ' is not right-veering with respect to C_k is even more obvious). We can see from the pictures that if $r_k < 0$ for some k, then $\phi'(\alpha_k)$ is to the left of α_k , so ϕ' is not right-veering which implies by Theorem 2.4 that $\xi_{\phi'}$ is overtwisted. Note that in any picture in Figure 8, we are taking all the other r_k's to be zero. However, even if ϕ' has a factor of some positive power of Dehn twist along the boundary component other than C_k, $\phi'(\alpha_k)$ is still left to the α_k at their common starting point by Lemma 3.1. Therefore, $\xi_{\phi'}$ is overtwisted by Corollary 3.2.

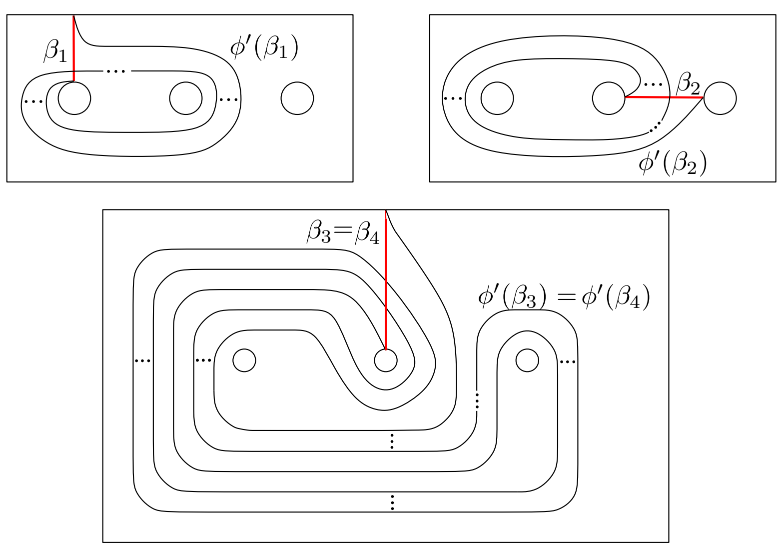

FIGURE 9. The curves β_k and their images under ϕ' in Σ.

To prove (OT2), consider the properly embedded curves $\beta_1, \beta_2, \beta_3, \beta_4$ starting at the boundary components C_1, C_2, C_3, C_4, respectively, and their images under ϕ' as given in Figure 9. In all the pictures, we are assuming $m = -1$, $n > 0$, (again otherwise the fact

that ϕ' is not right-veering with respect to C_k is even more obvious). We can see from the pictures that if $r_k = 0$ for some k, then $\phi'(\beta_k)$ is to the left of β_k , so ϕ' is not right-veering which implies again by Theorem 2.4 that $\xi_{\phi'}$ is overtwisted. Again, in all the pictures, we consider all the other r_k's to be zero, and if ϕ' has a factor of some positive power of Dehn twist along the boundary component other than C_k, $\phi'(\beta_k)$ is still left to the β_k at their common starting point by Lemma 3.1. Therefore, $\xi_{\phi'}$ is overtwisted by Corollary 3.2.

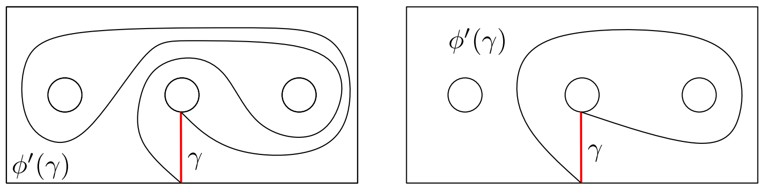

FIGURE 10. The curve γ and its images under two possible ϕ' in Σ.

To prove (OT3), consider the curve γ running from C_2 to C_4 as in Figure 10. In the left picture each $r_k = 1, m = -2, n = -1$, and in the right one each $r_k = 1, m = -1, n = -2$. Clearly, the image $\phi'(\gamma)$ is to left of γ at both their common endpoints on C_2 and C_4. Therefore, $\xi_{\phi'}$ ($\phi' = abcde^{-2}f^{-1}$ or $abcde^{-1}f^{-2}$) is overtwisted. In both cases, if we take r_1, r_3 and only one of r_2 and r_4 to be any positive integer, $\xi_{\phi'}$ is still overtwisted by Lemma 3.1 and Corollary 3.2. Moreover, if we also take $m \leq -3, n \leq -3$ in both cases, $\xi_{\phi'}$ is still overtwisted by Lemma 3.3 and Corollary 3.4.

The proof of (OT4) is similar to that of (OT3), so we will omit it. $\square$

The main trick we have used through out the paper is that we proved most of the the statements for $\phi' = a^{r_1}b^{r_2}c^{r_3}d^{r_4}e^m f^n$ and then applied Theorem 2.1. For the cases which we could not decide whether $\xi_{\phi'}$ is overtwisted or not, it is still good to know if ϕ' is right-veering. We give some partial answers to this in the next theorem where we do not list some obvious (or already proven) cases.

Theorem 4.3. $\phi' = a^{r_1}b^{r_2}c^{r_3}d^{r_4}e^m f^n \in Aut(\Sigma, \partial\Sigma)$ *is right-veering in the following cases:*

(R1) $min\{r_k\} = 1$, $mn = 0$, *and* $max\{m, n\} = 0$,

(R2) $min\{r_k\} = 1$, *and* $mn < 0$.

Proof. For (R1), assume $m < 0$ and $n = 0$. The fact that ϕ' is right-veering is an implication of Lemma 2.5 as follows: To show ϕ' is right-veering with respect to C_1 and C_2, take S' (in the lemma) to be the subsurface of Σ such that $\partial S' = C_1 \cup C_2 \cup e$ and take γ (in the lemma) as $\gamma = c^{r_3}d^{r_4}e^m$. To show ϕ' is right-veering with respect to C_3 and C_4, take S' to be the subsurface of Σ such that $\partial S' = C_3 \cup C_4 \cup e$ and take γ as $\gamma = a^{r_1}b^{r_2}e^m$.

For (R2), assume $m < 0$ and $n > 0$. We know, by (R1), that $\tilde{\phi} = a^{r_1}b^{r_2}c^{r_3}d^{r_4}e^m$ ($r_k \geq 1$ for all k, $m < 0$) is right-veering. Since $Dehn^+(\Sigma, \partial\Sigma) \subset Veer(\Sigma, \partial\Sigma)$, we conclude that $\phi' = \tilde{\phi} \cdot f^n$ ($n > 0$) is also right-veering. $\square$

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Department of Mathematics, University of Rochester, Rochester NY 14627, USA
E-mail address: `arikan@math.rochester.edu`

Department of Mathematics, Grand Valley State University, Grand Rapids MI 49401, USA
E-mail address: `durusoyd@gvsu.edu`